3 Reflections on Doing and Teaching Mathematics

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This two-part chapter is concerned with issues of mathematical philosophy an pedagogy. Part I deals with issues of ontology and/or epistemology—or in mor down-to-earth language, what it means to *do* mathematics. Part II, which i grounded in epistemological issues but focuses on issues of instruction, provide descriptions of selected aspects of my problem-solving courses. Those course are designed to engage students in the practices of doing mathematics and, as result, to have them develop a sense of discipline (i.e., a mathematical perspec tive) consistent with that held by mathematicians.

In a formal sense, Part I is neither necessary nor sufficient for Part II. On may do mathematics one way and teach it another, of course. Conversely, philos ophy can inform pedagogy but not determine it. Yet for most people there is a extremely strong relationship between Parts I and II. Whether or not one i explicit about it, one's epistemological stance serves to shape the classroon environments one creates (Hoffman, 1989). In turn, our classrooms are th primary source of mathematical experiences (as they perceive them) for ou students, the experiential base from which they abstract their sense of wha mathematics is all about. Hence, getting our epistemology straight, or at leas into the open for discussion, is a vitally important enterprise.

PART I: EPISTEMOLOGICAL ISSUES

The past few years have seen attempts on the part of philosophers and mathema ticians to reconceptualize and redescribe the mathematical enterprise. The recon ceptualization has its roots in the work of Pólya (1954), Lakatos (1977, 1978)

Benacerraf and Putnam (see, e.g., Benacerraf & Putnam, 1964), and more recently in the writings of Kitcher (1984). A main theme in that work is that the *doing* of mathematics is a (somewhat) empirical endeavor (see, e.g., Lakatos, 1978, pp. 30–34, "Mathematics is Quasi-empirical"). More pragmatically, mathematical authors such as Steen (1988) and Hoffman (1989) have tried to frame a popular notion of mathematics that accurately reflects the nature of contemporary mathematics and that also serves as a basis for a modern pedagogy of mathematics. The ideas expressed here are grounded in some philosophical reflections (discussed later), but they start off in a practical vein. Hoffman's ideas on the nature of mathematics and the need for reform in mathematics education are used as a starting point for discussion. The following are some of the main points made by Hoffman (1989):

A. The current system of mathematics education:
   - misrepresents mathematics, presenting it as a dead and deadly discipline;
   - is based on a false mastery model, in which isolated skills are taught in the hope they can then be used to solve prepackaged problems;
   - dumps by the wayside, after 8th grade, roughly 50% of the kids each year—with much higher percentages for most minorities;
   - is self-reproductive, in that the successes of the system are the ones who perpetuate it, and they have no models but the ones they've gone through;
   - is hence in need of comprehensive overhaul.

B. We need a powerful shorthand description of what mathematics is to convey the flavor of the discipline and to guide our teaching of it.
   - Mathematics is *the science of patterns*.

I agree with all of A, noting that there is widespread recognition of the problem and significant progress in working on it (see, e.g., California Department of Education, 1992; National Council of Teachers of Mathematics, 1989, 1991; National Research Council, 1989, 1990). Let me now turn to B.

I agree with Hoffman's arguments, as far as he takes them (see below for detail). Rather than as a descriptive end, however, I see the delineation of mathematics as "the science of patterns" as a point of departure. Describing mathematics that way raises some interesting issues and takes us into territory that Hoffman and others who have used the term (e.g., Steen, 1988) may not have anticipated. I'd like to venture into that territory.

Just What is Mathematics Anyway?

When mathematicians talk about mathematics, they usually mean the *products* of mathematics.

Hoffman (1989) begins with two questions, one metaphysical (What is mathematics?) and one epistemological (What does it mean to "know" mathematics?). The second question is a misdirection, albeit a subtle one. The question we should be asking instead is, "What does it mean to *do* mathematics, or to act mathematically?" The answer to this question comes from a liberal (well, perhaps radical) interpretation of Hoffman and Steen's answer to the first question, that mathematics is the science of patterns.

Steen's (1988) article on the mathematical enterprise is essential reading for everyone in the mathematics and mathematics education communities, including—perhaps most importantly—students. It describes the scope and depth of modern mathematics and its power in an increasingly mathematical world. Here are a few samples:

- Number Theory. "Fifty years ago G. H. Hardy could boast of number theory as the most pure and least useful part of mathematics; today number theory is studied as an essential prerequisite to many applications of coding, including data transmission from remote satellites" (p. 611).
- Applications. "The 1979 Novel prize in medicine was awarded to Allan Cormack for his application of the Radon transform, a well-known technique from classical analysis, to the development of tomography and computer assisted tomography (CAT) scanners. . . . Structural biologists have become genetic engineers, capturing the geometry of complex macromolecules in supercomputers and then simulating interaction with other molecules" (p. 614). Stochastic differential equations are now used to model chemical processes, stock market behavior, and population genetics.
- Core mathematics. The past 15 years have brought the solutions to some major unsolved problems, for example, the four-color problem and the Bieberbach and Mordell conjectures, and opened up new areas, such as prime factorization both of integers and polynomials.

Steen's main point is that old conceptions of mathematics were never quite accurate, and that they now fall short of the mark. Mathematics, classically defined as "the science of number and space" (Steen, 1988, p. 611) now includes the study of regularities of all sorts—not only in patterns of twin primes, but in the patterns emerging from CAT scans as well. Hence, he proposes, with Hoffman, that mathematics is the science of patterns.

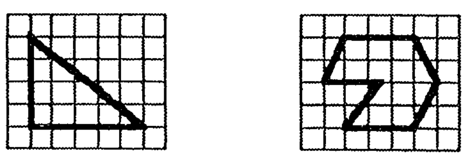
I am about to stretch the implications of this description. To set up the coming contrast, let me delineate what I think will be the standard interpretation of the phrase—a view that may be narrower than Steen's and Hoffman's. From the typical mathematician's point of view, *mathematics* is the "stuff" characterized above (number theory, applications, core math, etc.); *learning mathematics* is finding out about that stuff (often by being told, but sometimes by being presented with the opportunity to develop it on one's own); and *doing mathematics* is reaching the stage at which one is producing more of that stuff by oneself or in collaboration with others. Starting from "a science of patterns," a quite different view can be pursued.

A Brief Rhapsody on the *Science* of Patterns

The patterns part of the phrase requires no elaboration. Mathematics consists of observing and codifying—in general via abstract symbolic representations—regularities in the worlds of symbols and objects. (Work in these two spheres comprises pure and applied mathematics respectively.)

The science part is more interesting. To begin, a general (and positive) entailment of the term is that science is about making sense of things—finding out what makes them tick. From my point of view (see, e.g., Schoenfeld, 1987, 1990), that's precisely what mathematics is all about—a particular kind of sense-making, in which one's main tool kit consists of a set of symbolic tools, and there are well-established styles of reasoning for seeing how things fit together. Furthermore, "doing science" is generally recognized to be a social rather than a mere individual and solitary act. There is a scientific community that shares and builds ideas. So there is of necessity a premium on being able to communicate scientific results as well as on getting answers. It's that way in the mathematical community as well. To remind us that these are not the general perceptions regarding mathematics, let me briefly recall one shopworn example and introduce a fresher one. Consider these two problems:

1. An army bus holds 36 soldiers. If 1,128 soldiers are being bussed to their training site, how many buses are needed?
2. Imagine you are talking to a student in your class on the telephone and want the student to draw some figures. [They might be part of a homework assignment, for example.] The other student cannot see the figures. Write a set of directions so that the other student can draw the figures exactly as shown below.



Problem 1 comes from the Third National Assessment of Educational Progress (Carpenter, Lindquist, Matthews, & Silver, 1983). Seventy percent of the students who took the exam did the relevant computation correctly—and 29% of the students (41% of those who did the right calculation) went on to say that the number of buses needed is "31 remainder 12." Problem 2 comes from the 1987–88 California Assessment Program's statewide assessment of 12th graders' mathematical skills (California Department of Education, 1989). Only 15% of the high school seniors who worked the problem were able to describe the figures with any degree of clarity.

These are negative examples, which show in striking ways just how mathematics isn't learned—and hence point to what's missing in mathematics instruction. In the first case, you can't write down "31 remainder 12" if you are thinking about real buses. It's clear that for the students who wrote that answer, the problem wasn't about real objects at all. Many if not most students see mathematics word problems simply as cover stories that give rise to computations. Their learned behavior is that one does the computations and writes the answers down, period—never mind if the answer doesn't make sense outside that context. That's about as far from mathematics as sense-making as you can get. In the second case, the reason so few of the students could communicate about mathematics is very simple: They'd had little or no practice at doing so. When mathematics is taught as received knowledge rather than as something that (a) should fit together meaningfully, and (b) should be shared, students neither try to use it for sense-making nor develop a means of communicating with it.

These two examples represent just the tip of the iceberg, of course. Elsewhere (Schoenfeld, 1992, p. 359) I have written about student beliefs such as the following:

- Mathematics problems have one and only one right answer.
- There is only one correct way to solve any mathematics problem—usually the rule the teacher has most recently demonstrated to the class.
- Ordinary students cannot expect to understand mathematics; they expect simply to memorize it and apply what they have learned mechanically and without understanding.
- Mathematics is a solitary activity, done by individuals in isolation.
- Students who have understood the mathematics they have studied will be able to solve any assigned problem in 5 minutes or less.
- The mathematics learned in school has little or nothing to do with the real world (cf. the bussing problem).
- Formal proof is irrelevant to the processes of discovery or invention.

The roots of such beliefs reside, alas, in the students' classroom experience. But enough negativity; let me return to the theme of mathematics as the science of patterns.

Note that *hands on* and *empirical* (meaning "grounded in the results of data-gathering") are terms that at least *sound* natural with regard to science. Here is the official word, from Webster's *New Universal Unabridged Dictionary* (1979): "Science . . . systematized knowledge derived from observation, study, and experimentation carried on in order to determine the nature or principles of what is being studied."

In fact, that's precisely what I think mathematics is all about. The *result* of mathematical thinking may be a pristine gem, presented in elegant clarity as a polished product (e.g., as a published paper). Yet the path that leads to that product is most often anything but pristine, anything but a straightforward chain of logic from premises to conclusions.

Here is a generic description of the genesis of a mathematical result (e.g., a theorem) that takes a mathematician, say M, a few months to derive. Somewhere near the beginning of the process, M has the intuition that the result ought to be true, and thinks she knows why. So, she begins to sketch out a proof. Part of the argument goes fine, but then there is a place at which things bog down; she can't get an intermediate result that seems necessary. M tries three or four different ways of getting around the difficulty, without success. So, she begins to think the result might not be true. If not, there ought to be a counterexample—at the point in which she has run into trouble, of course. She tries to construct one, but it does not work. Nor does a second, a third, and so on, and then M sees that all the counterexamples fail for the same reason. That reason is the idea that has been missing from the proof, and M now gets past the roadblock. Of course, she encounters others as she continues working on the theorem. M is fortunate this time: The empirical data (attempts at counterexamples, etc.) work in her favor, and they result in her finding the ideas that allow for her proof. Other times she is less fortunate: Promising potential theorems turn out not to be true, and that's the end of the story.

In sort, mathematics is a "hands-on," data-based enterprise for those who engage in it. Doing mathematics is doing science, as defined above. It has a significant empirical component, one of data and discovery. What makes it mathematics rather than chemistry or physics or biology is the unique character of the objects being studied and the tools of the trade.

Another characteristic of the scientific enterprise is that it is, in large measure, a social enterprise. Many of the problems considered central are too big for people to solve in isolation. In consequence an increasingly large percentage of mathematical and scientific work is collaborative. Such collaborative work both requires and fosters shared perspectives, among collaborators in particular and across the field at large. When we say someone is a member of the scientific community, that phrase has significant entailments. It means that the person has the appropriate knowledge base, of course. But it also means that a person has picked up not only the tools but the perspectives of his or her discipline—a particular way of seeing the world, a style of thinking about it. (The stereotypes about doctors and lawyers, for example, do have a basis in reality; members of those groups tend to have particular ways of seeing the world. So do mathematicians, who develop their world views in the same way as do the others—by interacting with those who are already members of the community.)

I hope you are with me so far, because I am about to up the ante. The issue is the character of mathematical *knowing:* whether mathematicians can always be absolutely confident of the truth of certain complex mathematical results, or whether, in some cases, what is accepted as mathematical truth is in fact the best collective judgment of the community of mathematicians, which may turn out to be in error. I will argue the latter and will argue that taking this perspective has implications for classroom practice.

The notion of "fallible truths" and the role of the scientific community in defining those truths is more familiar in the case of the physical sciences. Popper (1959) and Kuhn (1962) highlighted the idea, and it has received a fair amount of recent discussion. The notion that absolute truth is unattainable in science is at least implicit in the language of science, in the use of the term *theory* for "tentative explanation." Again, thanks to Webster, a theory is "a formulation of apparent relationships or underlying principles of certain observed phenomena which has been verified to some degree." Basic science consists in large part of theory development and refinement, the construction of explanatory frameworks that account for data as well as possible. In that context, laws have a funny meaning. Scientists understand that the laws of science are not statements of absolute truth but merely theories that appear to have exceptionally solid grounding. New data, or different and more encompassing explanations, can result in the old laws losing credence and new versions taking their place (e.g., relativity supplants Newtonian mechanics, which supplanted the Aristotelian view).

Now the stereotype is that it's different in mathematics: It appears that you start with definitions or axioms and all the rest follows inexorably. However, as Lakatos (1977) shows in *Proofs and Refutations*, that isn't the way things really happen. The "natural" definition of polyhedron was accepted by the mathematical community for quite some time and was used to prove Euler's formula—until mathematicians found solids that met the definition but failed to satisfy the formula. How did the community deal with the issue? Ultimately, by changing the definition. That is, the grounds for the theory—the definitions underlying the system—were changed in response to the data. That sure looks like theory change to me: New formulations replace old ones, with base assumptions (definitions and axioms) evolving as the data come in. (In his 1978 work, *Mathematics, Science, and Epistemology,* Lakatos uses the term *quasi-empirical* to describe mathematics; he notes that mathematical theories cannot be true; they are at best "well-corroborated, but always conjectural" [p. 28].) What do we have then, regarding the nature of truth in mathematics (as in science)? To state things in the most provocative form: With regard to some very complex issues, truth in mathematics is that for which the vast majority of the community believes it has compelling arguments. And such truth may be fallible.

Serious mistakes are relatively rare, of course. For topics such as simple arithmetic or elementary real analysis, to pick two, there's no room for doubt. Once you make the definitions, the results follow—and the chain of logic that leads to the conclusions is sufficiently accessible so that anyone trained in the mathematics (i.e., who knows the rules of the game and plays by them) can confirm them. But, for complex results (e.g., a false proof of the Jordan Curve Theorem was widely known and accepted for a decade, and there was great controversy over the proofs of the four-color theorem and the Bieberbach conjecture), there is a social dimension to what is accepted as mathematical "truth." Once one accepts this notion, discussions of some traditional epistemological/ontological questions—questions of what it means to know mathematics and of mathematical authority (where does mathematical certainty reside?)—take on an interesting character. These are pursued in Part II.

Here is a distillation of my story so far: Mathematics is an inherently social activity, in which a community of trained practitioners (mathematical scientists) engages in the science of patterns—systematic attempts, based on observation, study, and experimentation, to determine the nature or principles of regularities in systems defined axiomatically or theoretically ("pure math") or models of systems abstracted from real-world objects ("applied math"). The tools of mathematics are abstraction, symbolic representation, and symbolic manipulation. However, being trained in the use of these tools no more means that one thinks mathematically than knowing how to use shop tools makes one a craftsperson. Learning to think mathematically means (a) developing a mathematical point of view—valuing the processes of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade and using those tools in the service of the goal of understanding structure—mathematical sense-making. Finally, some mathematical truths (results accepted as true by the community) are in fact "provisional truths," reflecting the field's best but possibly incorrect understanding.

The Bottom Line

Why raise all this fuss about the nature of mathematics? Because people develop their understanding of the nature of the mathematical enterprise from their experience with mathematics, and that experience (at least the part that is typically labeled as being "mathematics") takes place predominantly in our mathematics classrooms. The nature of that experience at present was described by Hoffman and summarized at the beginning of this chapter; the consequences of that experience are illustrated by the list of student beliefs summarized earlier in this section. When mathematics is taught as dry, disembodied, knowledge to be received, it is learned (and forgotten or not used) in that way. However, there is an optimistic counterpoint to the observation that one's experience with mathematics determines one's view of the discipline, and it has its own imperative: The activities in our mathematics classrooms can and must reflect and foster the understandings that we want students to develop with and about mathematics. That is, if we believe that doing mathematics is an act of sense-making; if we believe that mathematics is often a hands-on, empirical activity; if we believe that mathematical communication is important; if we believe that the mathematical community grapples with serious mathematical problems collaboratively, making tentative explanations of these phenomena, and then cycling back through those explanations (including definitions and postulates); if we believe that learning mathematics is empowering and that there is a mathematical way of thinking that has value and power, then our classroom practices must reflect these beliefs. Hence, we must work to construct learning environments in which students actively engage in the science of mathematical sense-making, as characterized earlier. Part II describes aspects of my attempts in that direction.

PART II: PEDAGOGICAL ISSUES

Elsewhere (see, e.g., Schoenfeld, 1985) I have characterized the mathematical content of my problem-solving courses. Here, in an extension of the themes explored in a number of recent (and one not-so-recent) papers (Balacheff, 1987; Collins, Brown, & Newman, 1989; Fawcett, 1938; Lampert, 1990; Lave, Smith, & Butler, 1988; Lave & Wenger, 1989; Schoenfeld, 1987, 1989b, 1992) I focus on the epistemological and social content and means. The content of my problem-solving courses is *epistemological* in that the courses reflect my epistemological goals: By virtue of participation in them, my students will develop a particular sense of the mathematical enterprise. The means are *social*, for the approach is grounded in the assumption that people develop their values and beliefs largely as a result of social interactions. I work to make my problem-solving courses serve as microcosms of selected aspects of mathematical practice and culture—so that by participating in that culture, students may come to understand the mathematical enterprise in a particular way.

What follows are two illustrations of goals, practices, and results. Those illustrations might be called protoethnographic. Though they might appear anecdotal, I believe they contain the substance from which good ethnographic descriptions could be crafted.

Example 1: Where Does Mathematical Authority Reside?

As indicated in Part I, mathematical truth or correctness is a delicately grasped object. One might say that the ultimate authority is the mathematics itself: False proofs are still false, even if people believe them, for example (and ultimately, one expects, the flaws in them will be uncovered). Nonetheless, mathematical authority is, in practice, exercised by human hands and minds. There are, of course, collective standards for mathematical correctness, for example, the review process, in which experts certify (to the degree they can; cf. the proof of the four-color theorem) that an argument is correct. Through such processes the mathematical community implements mathematical authority with consistency and (in general) with accuracy. This public process both is based in, and provides substance for, individuals' mathematical knowledge and authority. Mathematicians, having internalized the standards of correctness in their mathematical communities, apply those standards to what they know as individuals. In turn, the application of that abstract mathematical authority results in a very powerful personal ownership of the mathematics they can certify. To put it another way, arriving at mathematical certainty is the very personal process of applying an internalized impersonal standard.[1] In that sense, ultimate mathematical authority resides deeply in individuals, and collectively in the mathematical community.

Now, contrast this view of where authority resides with the typical student's view. Most college students possess little of the sense of personal knowledge or internal authority just described. They have little idea, much less confidence, that they can serve as arbiters of mathematical correctness, either individually or collectively. Indeed, for most students, arguments (or purported solutions) are merely proposed by themselves. Those arguments are then judged by experts, who determine their correctness. Authority and the means of implementing it are external to the students. Students *propose;* experts judge and *certify.*

One explicit goal of my problem-solving courses is to deflect inappropriate teacher authority. I hope to make it plain to the students that the mathematics speaks through all who have learned to employ it properly, and not just through the authority figure in front of the classroom. More explicitly, a goal of instruction is that the class becomes a community of mathematical judgment which, to the best of its ability, employs appropriate mathematical standards to judge the claims made before it.

In the course discussed here, the explicit deflection of teacher authority began the second day of class when a student volunteered to present a problem solution at the board. As often happens, the student focused his attention on me rather than on the class when he wrote his argument on the board; when he finished he waited for my approval or critique. Rather than provide it, however, I responded as follows:

> "Don't look to me for approval, because I'm not going to provide it. I'm sure the class knows more than enough to say whether what's on the board is right. So (turning to class) what do you folks think?"

In this particular case the student had made a claim that another student believed to be false. Rather than adjudicate, I pushed the discussion further: How could we know which student was correct? The discussion continued for some time, until we found a point of agreement for the whole class. The discussion proceeded from there. When the class was done (and satisfied) I summed up.

This problem discussion illustrated a number of important points for the

[1]One is likely to get to this point via interactions with others, of course.

students, points consistently emphasized in the weeks to come. First, I rarely *certified* results, but turned points of controversy back to the class for resolution. Second, the class was to accept little on faith. That is, "we proved it in Math 127" was not considered adequate reason to accept a statement's validity. Instead, the statement must be grounded in mathematics solidly understood by this class. Third, my role in class discussion would often be that of a Doubting Thomas. That is, I often asked, "Is that true? How do we know? Can you give me an example? A counterexample? A proof?," both when the students' suggestions were correct and when they were incorrect. (A fourth role was to ensure that the discussions were respectful—that it is the mathematics at stake in the conversations, not the students!)

This pattern was repeated consistently and deliberately, with effect. Late in the second week of class, a student who had just written a problem solution on the board started to turn to me for approval, and then stopped midstream. She looked at me with mock resignation and said, "I know, I know." She then turned to the class and said, "O.K., do you guys buy it or not?" [After some discussion, they did.]

The pattern continued through the semester. It was supplemented by overt reflections on our discussions that focused on what it means to have a compelling mathematical argument. The general tenor of these discussions followed the line of argumentation outlined in Mason, Burton, and Stacey's (1982) *Thinking Mathematically:* First, convince yourself; then, convince a friend; finally, convince an enemy. (That is, first make a plausible case and then buttress it against all possible counterarguments.) In short, we focused on what it means to truly understand, justify, and communicate mathematical ideas.

The results of these interactions revealed themselves most clearly in the following incident. Toward the end of the semester I assigned the following problem.[2]

The Concrete Wheel Problem

> You are sitting in a room at ground level, facing a floor-to-ceiling window which is 20-feet square. A huge solid concrete wheel, 100 miles in diameter, is rolling down the street and is about to pass right in front of the window, from left to right. The center of the wheel is moving to the right at 100 miles per hour. What does the view look like, from inside the room, as the wheel passes by? (See Fig. 3.1.)

This problem tends to provoke immediate and widely divergent intuitive reactions, among them:

1. The room will go (almost) instantaneously dark as the wheel first passes the window. It will stay dark for a short while and go (almost) instantaneously light as the wheel leaves.

[2]The problem is borrowed from diSessa (and borrowed in turn, I believe, from Papert).

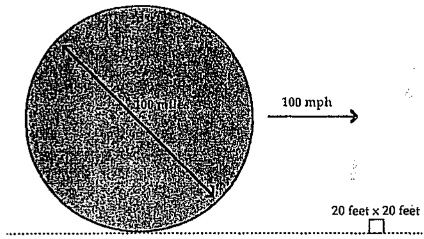


FIG. 3.1. The situation described in the concrete wheel problem.

2. Same as (1), but the room stays dark for a relatively long time.
3. The room darkens slowly, as though a large window shade was being pulled more or less:
   a. horizontally from left to right, as follows:

   b. diagonally from the upper left corner as follows:

   c. vertically downward as follows:

   The room then stays dark for a short/long period of time, after which it lightens in a way complementary to the way it darkened.

When the problem was posed, students made many of the conjectures listed above. As usual, the class broke into groups to work on the problem. One group became the staunch defenders of one conjecture, while a second group lobbied for another. The two groups argued somewhat heatedly, with the rest of the class following the discussion. Finally, one group prevailed, on what struck me as solid mathematical grounds.[3] As is my habit, I did not reveal this but made my usual comment: "O.K., you seem to have done as much with this as you can. Shall I try to pull things together?" One of the students replied, "Don't bother. We got it." The class agreed.

While one might dismiss this event as being trivial (the students simply indicated that they had understood the material, and the class progressed; what's

[3] I shall refrain from giving the answer in order not to spoil readers' possible pleasure in determining it themselves.

the big deal?), or even see their rejection of my offer to pull things together as being somewhat abrasive (I had signaled my intentions, and they told me not to bother), either view misses the significance of the event. First, it is important to note that the classroom was functioning as a mathematical community. Various points of view were advanced and defended mathematically. The arguments in favor of different positions were made on solid mathematical grounds, and ultimately the correct view prevailed, for good reason. One could ask for no better at a meeting of professional mathematicians. Second, and more importantly, the locus of mathematical authority had shifted radically. From the student's point of view, I was no longer needed as an authority figure to provide external *certification* of results. As in the mathematical community at large, the mathematics spoke through the students. It did so collectively, in the dialogue that took place in the community; it did so individually, in that the students demanded the appropriate mathematical standard of argumentation, and then believed the results. This was *their* mathematics. They had ownership of it, not only in the motivational sense, but in the deep epistemological sense that characterizes the true mathematical knowing and understanding possessed by mathematicians.

Example 2: Who Can *Do* Mathematics?

Speaking broadly, research mathematics is one thing, and classroom mathematics is something else altogether. When they are *doing* mathematics as researchers, mathematicians are pushing the boundaries of knowledge—not only their own, but that of the mathematical community. Publishable research consists, in essence, of results that (a) are new to the community of mathematicians, and (b) deemed of sufficient merit or interest to warrant distribution. In contrast, classroom mathematics generally consists of the distillation and presentation of known results to be "mastered" by students. The implicit but widespread presumption in the mathematical community is that an extensive background is required before one can *do* mathematics. Undergraduates who publish mathematics are exceedingly rare, and even graduate students with publications prior to their thesis work are relatively uncommon. Until students get to the point of doing research (typically in the third year of graduate school), learning mathematics means *ingesting* mathematics.

There are, of course, exceptions to this rule. There is, for example, the Moore method. The *Journal of Undergraduate Research* has, for half a century, published student work in mathematics. Occasionally classroom work produces results of professional quality. For example, an article by Banchoff and student associates (1989) appeared in a recent *UME Trends*, and a discussion in one of my problem-solving classes not long ago led to a publication in the *College Mathematics Journal* (Schoenfeld, 1989a). But the threshold of research *qua* research is unreasonably high for most undergraduate courses.

Here I wish to pursue an alternative perspective, one based on the notion of *intellectual community.* In the introduction to this chapter I indicated that I work to make my problem-solving courses "microcosms of selected aspects of mathematical practice and culture," in that the classroom practices reflect (some of) the values of the mathematical community at large.

Part I of this chapter presented one mainstream view of mathematics, as the *science of patterns*. I described this elsewhere by saying that the business (and pleasure) of mathematics consists of perceiving and delineating structural relationships. Suppose we add to that the notion, as suggested above, that research—what most mathematicians would call *doing* mathematics—consists of making contributions to the mathematical community's knowledge store. And finally, one adds part of the mathematician's aesthetic, that making such contributions is part of the mathematician's intellectual life, and something of intrinsic value.

My goals for my problem-solving courses are to create local intellectual communities with those same values and perspectives. The notion of localization works as follows: A contribution is significant if it helps the particular intellectual community advance its understanding in important ways.

Elsewhere I (Schoenfeld, 1990) described one of my class's discussions of the Pythagorean Theorem. Here I review that discussion from the perspective of social and epistemological engineering. The initial problem posed to the class was very broad, in essence: "What can we do with the Pythagorean Theorem?"

In its discussion of the result (well known to all of the students), the class began by proving the theorem a variety of different ways. It explored three- and n-dimensional analogues of the theorem; it pursued geometric extensions and analogues. Then it began to focus on the diophantine equation

$$a^2 + b^2 = c^2.$$

Could we find all positive integer solutions to this equation?

Now, any mathematician can tell you there is a general solution to this problem. A triple of integers (a, b, c) with the property that $a^2 + b^2 = c^2$ is called a Pythagorean triple. Every Pythagorean triple (a, b, c) can be shown to be of the form

$$a = k(M^2 - N^2),\ b = 2k(MN),\ c = k(M^2 + N^2),$$

where k is the largest common factor of a, b, and c, and M and N are relatively prime integers. In a content-oriented course (e.g., elementary number theory), one would typically present the proof of this result in about 10 minutes, and then move on to another result. But part of the engineering effort in teaching this course consists of seeding classroom dialogue with problems at the appropriate level for community discourse, and then holding back as the community grapples with those problems to the best of its ability.

In this case, the students began working on the problem by generating some of the whole-number Pythagorean triples they knew; (3, 4, 5), (5, 12, 13), (6, 8, 10), (7, 24, 25), (8, 15, 17), (9, 40, 41), and (10, 24, 26). On the basis of these empirical data they made the following observations:

1. Integer multiples of Pythagorean triples are Pythagorean and hence of little intrinsic interest. (If you can generate all the relatively prime Pythagorean triples, then you can generate all the rest.)
2. In every relatively prime triple, the hypotenuse was odd.
3. In every relatively prime triple where the smaller leg was odd, the hypotenuse exceeded the larger leg by 1.
4. In the relatively prime triple where the smaller leg was even, the hypotenuse exceeded the larger leg by 2.

As a result of observation (1), the class restricted its attention to relatively prime Pythagorean triples. They conjectured that observation (2) was always true, and proved it. On the basis of observation (3), they conjectured that there are infinitely many Pythagorean triples of the form $(2x + 1, 2y, 2y + 1)$, and proved it. On the basis of observation (4), they conjectured that there are infinitely many Pythagorean triples of the form $(2x, 2y - 1, 2y + 1)$, and proved it. On the basis of those two results, and the fact that they knew of no other triples, the class conjectured that all relatively prime triples are of the types described in (3) and (4). They began their work on this conjecture by proving it for the first relevant case: They proved there are no relatively prime Pythagorean triples of the form $(x, y, y + 3)$. At that point a student asked: If they were successful in proving their conjecture, did they have a publishable theorem?

The answer, of course, was no. As noted above, the complete solution to the problem they were working on is a standard result presented in elementary number theory courses. Nonetheless, neither the student's question nor the class's achievements should be discounted. The individual student's comment indicated that he, at least, thought that the class might be at the frontiers of knowledge—a far cry from what happens in most classrooms.

And, in two significant ways, the students were. First, three of the results they proved:

> There are infinitely many triples of the form $(2x + 1, 2y, 2y + 1)$;
> There are infinitely many triples of the form $(2x, 2y - 1, 2y + 1)$; and
> There are *no* relatively prime triples of the form $(x, y, y + 3)$,

were new to me and (although easily proven) are a surprise to many mathematicians. Hence, the product of their labors was not inconsequential. But more importantly, these students, in their own intellectual community, were *doing* mathematics. They were, at a level commensurate with their knowledge and abilities, truly engaged in the science of patterns.

FINAL COMMENTARY

In Part I of this chapter I tried to portray mathematics as a living, breathing discipline in which truth (as much as we can know it) lives in part through the individual and collective judgments of members of the mathematical community. I suggested that

1. Mathematicians develop much of that deep mathematical understanding by virtue of apprenticeship into that community—typically in graduate school and as young professionals.
2. In standard instruction students are typically deprived of such apprenticeships, and hence of access to doing and knowing mathematics.

In Part II of this chapter I tried to convey some of the character of my problem-solving courses. In essence, I create artificial communities in them—communities in which certain mathematical values, consistent with some of those of the mathematical community at large, predominate. The following is a more precise delineation of some of the main themes of those courses:

1. Mathematics is the science of patterns, and relevant mathematical activities—looking to perceive structure, seeing connections, capturing patterns symbolically, conjecturing and proving, and abstracting and generalizing—all are valued.
2. Mathematical authority resides in the mathematics, which—once we learn how to heed it—can speak through each of us and give us personal access to mathematical truth. In that way mathematics is a fundamentally human (and for some, aesthetic and pleasurable) activity.

I hope to have illustrated, in examples 1 and 2, how students, by living in such artificial microcosms of mathematical practice, come to develop as mathematical doers and thinkers. I conclude this chapter with the comment that in a very serious sense, these artificial environments provide students with a genuine experience of *real* mathematics. By that standard, conventional mathematics instruction is wholly artificial.

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